stichting
mathematisch
centrum

MC

AFDELING ZUIVERE WISKUNDE
(DEPARTMENT OF PURE MATHEMATICS)

ZW 159/81 APRIL

A.M. COHEN

ON THE POINTS AND LINES OF METASYMPLECTIC SPACES

Preprint

kruislaan 413 1098 SJ amsterdam

Printed at the Mathematical Centre, 413 Kruislaan, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

1980 Mathematics subject classification: 51B25

On the points and lines of metasymplectic spaces*)

by

Arjeh M. Cohen

ABSTRACT

For metasymplectic spaces whose lines have at least three points, an axiom system is given that characterizes them in terms of points and lines.

KEY WORDS & PHRASES: *Metasymplectic spaces, buildings of type F_4, polar spaces*

*) This report will be submitted for publication elsewhere.

CONTENTS

1. TERMINOLOGY AND NOTATIONS

An *incidence system* (P,L) is a set P of *points* together with a collection L of subsets of cardinality > 1, called *lines*. If (P,L) is an incidence system then the point graph or collinearity graph of (P,L) is the graph (P,Γ) whose vertex set is P and whose edges consist of the pairs of collinear points. The incidence system is called *connected* whenever its collinearity graph is connected. Likewise terms such as *(co)cliques*, *paths* will be applied freely to (P,L) when in fact they are meant for (P,Γ). We let $d(x,y)$ for $x,y \in P$ denote the ordinary distance in (P,Γ) and write

$$\Gamma_i(x) = \{y \in P \mid d(x,y) = i\}.$$

Also

$$\Gamma(x) = \Gamma_1(x) \quad \text{and} \quad x^\perp = \{x\} \cup \Gamma(x).$$

For a subset X of P we write

$$X^\perp = \bigcap_{x \in X} x^\perp .$$

If $z \in P$ and X,Y are subsets of P, then

$$d(z,X) = \inf_{x \in X} d(z,x) \quad \text{and} \quad d(X,Y) = \inf_{y \in Y} (y,X).$$

A subset X of P is called a *subspace* of (P,L) whenever each point of P on a line bearing two distinct points of X is itself in X. A subspace X is called *singular* whenever it induced a clique in (P,Γ). The length i of a longest chain $X_0 \subsetneqq X_1 \subsetneqq \dots \subsetneqq X_i = X$ of nonempty singular subspaces X_j of X is called the *rank* of X.

For a subset X of P, the subspace generated by X is denoted $\langle X \rangle$. Note that it is well defined as P is a subspace containing X. Furthermore, $L(X)$ denotes the set of lines contained in X. The incidence system (P,L) is called *linear* if any two distinct points are on at most one line. If x,y are collinear points of a linear incidence system, then xy denotes the unique line through them. Thus $xy = \langle x,y \rangle$, where $\langle x,y \rangle$ is short for $\langle \{x,y\} \rangle$.

If F is a family of subsets of P and $x \in P$, then F_x denotes the subfamily of F of members containing x. Also, $L(F)$ denotes the collection of $L(X)$ for $X \in F$. Finally, a line is called *thick* if there are at least three points on it.

2. INTRODUCTION AND STATEMENT OF THE THEOREM

Metasymplectic spaces were introduced by H. FREUDENTHAL [6] as part of the study of geometries of type F_4. A set of axioms together with a characterization in terms of buildings of type F_4 under mild additional conditions, can be found in Tits' work [8,pp.215-217] . Our starting point will be the definition given there. In order to state the definition, however, we need the notion of polar spaces. For a definition of polar spaces of rank n we could cite [1] or [8]. Instead, we shall formulate the celebrated Buekenhout-Shult theorem from [1] for two reasons: i) it can be used as an alternative definition, and ii) it suffices for our purposes.

2.1. <u>THEOREM</u> (Buekenhout-Shult). *Let* (P,L) *be an incidence system and let* n *be a natural number. Then* P *together with its singular subspaces is a polar space of rank* $\leq$ n *if and only if the following holds:*

(P1) *No point of* P *is collinear with all of* P.

(P2) *For each point* x *of* P *and line* ℓ *of* L *with* $x \notin \ell$, x *is collinear with either one or all points of* ℓ.

(P3) *The singular subspaces of* P *are of rank* $\leq$ n.

It is well known that the extra hypothesis in [1] that all lines be thick is superfluous. Therefore, it is omitted here.

The theorem can be used as a definition of *a polar space of rank* $\leq$ n. Of course, a *polar space of rank* n is a polar space of rank $\leq$ n which is not a polar space of rank $\leq$ n-1. Polar spaces of rank 2 are called *generalized quadrangles*. This means that all generalized quadrangles are nondegenerate (i.e. axiom P1 holds). We shall often abuse language and call incidence systems polar spaces when in fact the associated geometry of its points and all its singular subspaces is a polar space. Similarly for projective spaces. The singular subspaces of polar spaces have the structure of projective spaces.

2.2. DEFINITIONS. A *metasymplectic space* is a set P in which some subsets, called *lines, planes* and *symplecta* are distinguished, and satisfy the following axioms:

(M1) *The intersection of two distinct symplecta is empty, or is a point, a line or a plane.*

(M2) *A symplecton, together with the "linear subspaces", empty set, points, lines and planes contained in it, is a polar space of rank* 3.

(M3) *Considering the set* x^* *of all symplecta containing a given point* x *of* P, *and calling lines (resp. planes) of* x^* *the subsets of* x^* *consisting of all symplecta containing a plane (resp. a line) through* x, *one also obtains a polar space of rank* 3.

A classical result of Veblen and Young characterizes projective spaces in terms of points and lines. The Buekenhout-Shult Theorem is a characterization of polar spaces in terms of points and lines. More recently, CAMERON [2] and SHULT-YANUSHKA [7] obtained a characterization of reverse (or dual) polar spaces in terms of incidence systems. Also Cooperstein's work [5] provides characterizations of this kind of certain geometries associated with buildings of Lie type. We shall make use of some of these results (see Section 3) to prove the following characterization of metasymplectic spaces in terms of points and lines.

2.3. THEOREM. *Let* (P,L) *be a connected incidence system. Then* P *and* L *can be identified with the points and lines of a connected metasymplectic space with thick lines or a polar space of rank* 3 *with thick lines if and only if* (P,L) *satisfies the following axioms:*

(F1) *For each* $x \in P$ *and each* $\ell \in L$ *either* 0, 1 *or all points of* ℓ *are collinear with* x.

(F2) *For each pair* $x,y \in P$ *with* $x \in y^\perp$, *the graph on* $x^\perp \cap y^\perp$ *is not a clique.*

(F3) *For each pair* $x,y \in P$ *with* $x \notin y^\perp$, *such that* $x^\perp \cap y^\perp$ *contains at least two points,* $x^\perp \cap y^\perp$ *together with the lines whose points are all in* $x^\perp \cap y^\perp$ *is a generalized quadrangle with thick lines.*

(F4) *There are no minimal* 5-*circuits: i.e. given* $x_1,x_2,x_3,x_4,x_5 \in P$ *with* $x_i \in \Gamma_1(x_{i+1})\backslash x_{i+2}^\perp$ *for each* i *(indices taken* mod 5*), there is an* i *for which* x_i *is collinear with a point on a line through* x_{i+2} *and* x_{i+3}.

(F5) *If* $x,y,z \in P$ *are such that* $x^\perp \cap y^\perp$ *has at least two points and* $y \in z^\perp$*, then* $x^\perp \cap z^\perp \neq \emptyset$.

To facilitate notations, a connected incidence system satisfying (F1),...,(F5) is called a *metapolar space*. The theorem states that the metapolar spaces of diameter 3 are precisely the connected metasymplectic spaces with thick lines (whose planes and symplecta are discarded).

2.4. REMARK. *An incidence system satisfying* (F1), (F2), (F3) *is linear and completely determined by its collinearity graph.* For, if (P,L) is such a system and $x \in P$, $y \in \Gamma_1(x)$, then there are $u,v \in P$ with $u \notin v^\perp$ such that $x,y \in u^\perp \cap v^\perp$ by (F2), so that $u^\perp \cap v^\perp$ is a generalized quadrangle by (F3). If ℓ is a line on x,y, then all its points belong to $u^\perp \cap v^\perp$ according to (F1), so that $\ell \subseteq u^\perp \cap v^\perp$ and $\ell^{\perp\perp} \subseteq \{u,v\}^{\perp\perp\perp} = u^\perp \cap v^\perp$. Thus $\ell^{\perp\perp}$ is a clique containing ℓ and contained in a generalized quadrangle and therefore equal to ℓ. But $\ell^\perp \subseteq \{x,y\}^\perp \subseteq \ell^\perp$ (again, using (F2)) whence $\ell = \ell^{\perp\perp} = \{x,y\}^{\perp\perp}$. This settles the remark.

We shall now briefly discuss the individual axioms. Axiom (F1) states that (P,L) is a Gamma space in D.G. Higman's terminology. It is known [4], that the points and lines in a metasymplectic space arising from a building of type F_4 satisfy this property.

Axiom (F2) is a nondegeneracy condition. Together with (F1), (F3) it ensures that the incidence system is in fact determined by its collinearity graph (as we have just seen) and that there do indeed exist generalized quadrangles. Generalized hexagons but also near octagons without 4-circuits (see [3]) are examples of incidence systems satisfying all axioms (F1),...,(F5) except for (F2).

Axiom (F3) provides the building blocks for symplecta. It may be regarded as a slight extension of Cooperstein's axiom 1.4 of [5].

Axiom (F4) is a weakened version of the near n-gon property (of [7]) that for any $x \in P$ and $\ell \in L$ with $d(x,\ell) = 2$ there is a unique point on ℓ nearest x. The geometries described by Cooperstein in Theorem B of [5] are examples of incidence systems in which all other axioms hold and which are not metasymplectic spaces.

Axiom (F5) may be viewed as another nondegeneracy condition. For it rules out bouquets of polar spaces, i.e. geometries (P,L) containing a point x such that any connected component of $P\backslash\{x\}$ joined with $\{x\}$ forms a subspace which is a polar space. Clearly, these bouquets satisfy all other axioms.

The proof of the theorem consists of Sections 4 and 5.

To conclude this introduction we give three more axioms which hold in metapolar spaces. It might be of interest to know to what extent they could replace axioms in the theorem.

(F6) *For* $x \in P$ *and* $\ell \in L$ *either* 1 *or all points* y *of* ℓ *satisfy* $d(x,y) = d(x,\ell)$.

(F7) *For* $\ell, m \in L$ *either* 1 *or all points* x *of* ℓ *satisfy* $d(x,m) = d(\ell,m)$.

(F8) *For* $x \in P$, $\ell \in L$ *with* $d(x,\ell) = 2$, *there is at most one point* y *in* ℓ *for which* $x^\perp \cap y^\perp$ *contains at least two points.*

The proofs of (F6), (F7) for metapolar spaces are omitted as they are immediate consequences of the results in Section 4. Axiom (F8) is shown to hold in 4.4.

3. PRELIMINARY RESULTS

Most of the material in this section is taken from the sources [2], [5] and [7]. The first lemma is a slightly altered version of Proposition 2.5 in [7], to which the reader is referred for a proof.

3.1. LEMMA (Shult-Yanushka). *Let* (P,L) *be an incidence system with thick lines such that for* $x \in P$, $\ell \in L$ *with* $d(x,\ell) \leq 2$, *there is a unique point of* ℓ *nearest* x. *Then* (P,L) *is linear. Moreover, if* a_1,a_2,a_3,a_4 *is a minimal 4-circuit in* P *(i.e.* $a_i \in \Gamma(a_{i+1})\backslash a_{i+2}^\perp$ *for all* i, *indices taken* mod 4*), then*

$$Q = \{x \in P \mid d(x,a_ia_{i+1}) \leq 1 \textit{ for each } i\}$$

is a subspace of P *which, together with the lines it contains, is a generalized quadrangle such that for any* $x \in P\backslash Q$, *the intersection* $x^\perp \cap Q$ *contains at most one point.*

3.2. DEFINITIONS. Let (P,L) satisfy the hypotheses of the above lemma. The subspaces X obtained from minimal 4-circuits as above are called *quads*. Two points x,y of mutual distance 2 are contained in at most one quad, denoted by $Q(x,y)$. Hence the intersection of two quads is either empty or a singular subspace.

If (P,L) is a connected incidence system, it is called a *near hexagon* (cf. [7]) whenever it satisfies the following axioms:

(N1) *For each* $x \in P$ *and* $\ell \in L$ *there is a unique point of* ℓ *nearest* x.

(N2) *The diameter of* (P,Γ) *is at most* 3.

The near hexagon (P,L) is called *classical* if it has thick lines and also satisfies:

(N3) *For each* $x \in P$, $y \in \Gamma_2(x)$, *the set* $x^\perp \cap y^\perp$ *contains at least two points.*

(N4) *For each quad* Q *and* $x \in P\backslash Q$, *the intersection* $x^\perp \cap Q$ *is nonempty.*

3.3. LEMMA. *Let* (P,L) *be a connected incidence system with thick lines such that for* $x \in P$, $\ell \in L$ *with* $d(x,\ell) \le 2$, *there is a unique point of* ℓ *nearest* x, *and suppose* (N3) *holds for* (P,L). *Then* (P,L) *is a classical near hexagon if and only if the intersection of no two quads is a point. Moreover, if this is the case, then:*

(N5) *For each quad* Q *and* $x \in P\backslash Q$ *the intersection* $x^\perp \cap Q$ *is a singleton.*

(N6) *Given a quad* Q *and distinct collinear* $x,y \in P$ *for which there are non-collinear* $u,v \in Q$ *satisfying* $v \in x^\perp$ *and* $u \in y^\perp$, *then* $xy \subseteq Q$.

PROOF. We shall leave the proof of (N5) and (N6) for a classical near hexagon to the reader.

If (P,L) is a classical near hexagon and Q,R are distinct quads with $x \in Q \cap R$ for some $x \in P$, we need to verify that $Q \cap R$ contains another point. Take $z \in \Gamma_2(x) \cap R$. Then $z \notin Q$ as $Q \cap R$ is a clique. In view of (N4) there is $y \in P$ with $y \in z^\perp \cap Q$. Note that $x \neq y$. If $y \in x^\perp$, then $y \in x^\perp \cap z^\perp \subseteq R$, so $Q \cap R$ contains x,y as wanted. On the other hand, $y \notin x^\perp$ implies $d(x,y) = 2 = d(x,z)$, so that there is $w \in yz \cap x^\perp$ by (N1). Now $w \in (x^\perp \cap z^\perp) \cap (x^\perp \cap y^\perp) \subseteq Q \cap R$ and we are done since $w \neq x$.

To prove the converse, let (P,L) be an incidence system satisfying the hypothesis and assume that the intersection of no two quads is a point. Let Q be a quad and let $x \in P\backslash Q$, $y \in x^\perp \cap Q$. Take $z \in x^\perp\backslash xy$ and consider the quad $Q(y,z)$ whose existence is guaranteed by (N3) and Lemma 3.1. Since

$y \in Q \cap Q(y,z)$, there must be a line m such that $Q \cap Q(y,z) = m$ and there is $w \in m \cap z^{\perp} \subseteq Q \cap z^{\perp}$. By induction with respect to the length of a path from an arbitrary point z of $P \backslash Q$ to a point of Q, we get $Q \cap z^{\perp} \neq \emptyset$ for any $z \in P \backslash Q$, proving (N4).

It remains to show that for $z \in P$ and $\ell \in L$ we have $d(z,\ell) \leq 2$. We may assume that the diameter of (P,L) is ≥ 3. Thus there exists a quad R containing ℓ. Note that for $z \in R$, we have $d(z,\ell) \leq 1$. Hence for arbitrary $z \in P$, axiom (N4) yields $d(z,\ell) \leq d(z,R) + 1 \leq 2$. This finishes the proof. □

The significance of classical near hexagons will become clear from the following result, which is a consequence of Theorem 1 in [2] reformulated in the terminology of near hexagons [7].

3.4. THEOREM (Cameron, Shult-Yanushka). *Let* (P,L) *be a connected incidence system with thick lines. Then* (P,L) *is a generalized quadrangle if and only if it is a classical near hexagon of diameter* 2. *Moreover,* P *can be identified with the maximal singular subspaces of a polar space of rank* 3 *in such a way that any line is the set of such subspaces containing a line of the polar space (and conversely) if and only if* (P,L) *is a classical near hexagon of diameter* 3.

We are now in a position to reformulate axiom (M3) in terms of near hexagons. Recall that if F is a collection of subsets of P, then $L(F_x)_x$ for $x \in P$ denotes the collection of subsets of lines through x contained in a member of F on x.

3.5. COROLLARY. *Let* (P,L) *be an incidence system with thick lines in which collections* V,S *of subsets of* P *are distinguished. Then* (P,L) *together with* V *for its collection of planes and* S *for its collection of symplecta, is a metasymplectic space if and only if* (M1), (M2) *and* $(M3)_3$ *hold, where* $(M3)_d$ *for* $d = 2,3$ *is the following axiom:*

$(M3)_d$ *For any* $x \in P$, $(L_x, L(V_x)_x)$ *is a classical near hexagon of diameter* d *whose quads are the members of* $L(S_x)_x$.

In the remainder of this section we recall some properties of incidence systems satisfying (F1), (F2) and (F3). They can be found in Section 3 of [5], where instead of (F3) a slightly stronger statement holds. However, the proofs are still valid in our case, so we shall not bother to copy them.

3.6. LEMMA. *Let* (P,L) *be an incidence system for which* (F1) *holds. Then*
(i) *maximal cliques are singular subspaces;*
(ii) *for any clique* X *of* P, *the subspace* $\langle X\rangle$ *is singular;*
(iii) *if* X *is a subset of* P, *then* $X^{\perp}$ *is a subspace.*

3.7. LEMMA. *Suppose* (P,L) *is an incidence system for which* (F1), (F2), (F3) *hold. Then*
(i) *if* $\ell,m \in L$ *with* $d(\ell,m) = 1$, *then* $d(y,m) = 1$ *for exactly* 1 *or all points* y *of* ℓ;
(ii) *if* $d(x,y) = 2$ *and* $x^{\perp} \cap y^{\perp}$ *contains at least two points, then* $\{x,y\}^{\perp\perp}$ *is a coclique;*
(iii) *if* x,y,z *form a clique in* P *not contained in a line, then* $\{x,y,z\}^{\perp}$ *is a maximal singular subspace.*

3.8. PROPOSITION (Cooperstein). *Let* (P,L) *be an incidence system satisfying* (F1), (F2), (F3). *If* $x,y \in P$ *with* $d(x,y) = 2$ *have at least two common neighbors, the subset* $S(x,y)$ *of all points* $z \in P$ *satisfying* $z^{\perp} \cap \ell \neq \emptyset$ *for each line* ℓ *of* $x^{\perp} \cap y^{\perp}$ *is a subspace which is a polar space of rank* 3. *Moreover,* $S(x,y) = \langle\{x\} \cup \{y\} \cup \{x,y\}^{\perp}\rangle$, *provided all lines are thick.*

3.9. DEFINITIONS. Let (P,L) be as in the preceding proposition. Any pair $x,y \in P$ with $d(x,y) = 2$ such that $x^{\perp} \cap y^{\perp}$ contains at least two points will be called a *symp pair*. The subspace $S(x,y)$ from above for such x,y will be called the *symp* on x,y and will be denoted as in the proposition.

The following corollary shows that the symp on x,y for $x \notin y^{\perp}$ is the unique symp containing x,y.

3.10. COROLLARY. *If* (P,L) *is an incidence system in which* (F1),(F2),(F3) *hold, and if* S,T *are symps in* P, *then* $S \cap T$ *is a singular subspace.*

3.11. LEMMA. *Let* (P,L) *be as in Proposition* 3.8.
(i) *Each singular subspace of rank* ≤ 2 *is contained in a symp. Hence it is a point, a line or a projective plane;*

(ii) *if* M *is a singular subspace and* M *properly contains a line, then* M *is a projective space.*

4. METAPOLAR SPACES ARE EITHER POLAR OR METASYMPLECTIC

Throughout this section (P,L) is a metapolar space. Let V be the collection of maximal singular subspaces. Their members are called *planes*. This name will be justified by Lemma 4.2. Furthermore, write S for the collection of symps. We shall show that (P,L) with points, lines, planes and S for the collection of symplecta satisfies axioms (M1), (M2) and $(M3)_d$ for $d = 2$ or 3. First of all, note that (M2) follows from Proposition 3.8.

4.1. LEMMA. *Let* x_1,x_2,x_3,x_4,x_5 *be a 5-circuit in* P *with* $x_i \in \Gamma_1(x_{i+1})\backslash x_{i+2}^{\perp}$ *(indices* mod 5*) for all* i. *Then there is a symp* S *containing all* x_i.

PROOF. By (F4), there is i, say $i = 1$, and $w \in P$ with $\{w\} = \Gamma(x_1) \cap x_3x_4$. Note that $w \neq x_3,x_4$. If $w \in x_2^{\perp}$, then $x_2 \in (x_3w)^{\perp}$ by (F1), and $x_2 \in x_4^{\perp}$, contradiction. Hence $w \notin x_2^{\perp}$; similarly $w \notin x_5^{\perp}$. But $x_1,x_3 \in x_2^{\perp} \cap w^{\perp}$, so x_2,w is a symp pair and $x_4 \in wx_3 \subseteq S(x_2,w)$ in view of Proposition 3.8. Thus $x_1,x_4 \in S(x_2,w)$ and $S(x_1,x_4) = S(x_2,w)$ as $d(x_1,x_4) = 2$ by the same proposition. The conclusion is that all x_i are in $S(x_2,w)$. □

4.2. LEMMA. *Maximal singular subspaces of* P *are projective spaces.*

PROOF. Axiom (F2) yields that lines are not maximal cliques. Let M be a maximal clique and suppose ℓ,m are lines in M with $\ell \cap m = \emptyset$. We shall derive a contradiction. Take $a \in \ell$ and $b \in m$ and let $x \in \ell^{\perp}\backslash b^{\perp}$, $y \in m^{\perp}\backslash a^{\perp}$ (such x,y exist according to (F2)). Then $x^{\perp} \cap M = \ell$ and $y^{\perp} \cap M = m$; for if $z \in x^{\perp} \cap M$, then $z \in x^{\perp} \cap b^{\perp}$ which is then a generalized quadrangle containing ℓ,z with $z \in \ell^{\perp}$, so that $z \in \ell$ (the reasoning for $y^{\perp} \cap M = m$ is similar). Now axiom (F5) applied to y,a,x yields $d(x,y) \leq 2$. If $x = y$, then $m = y^{\perp} \cap M \supseteq \ell \cup m$, so $\ell = m$ which is absurd. If $d(x,y) = 1$, then $b,x \in \{a,y\}^{\perp} \subseteq S(a,y)$ and $d(b,x) = 2$ so that $S(a,y) = S(b,x)$ and $\ell,m \subseteq S(a,y)$. Thus $\langle \ell,m\rangle \subseteq S(a,y)$. But then $\langle \ell,m\rangle$ is a singular subspace (Lemma 3.6(ii)) of a polar space which is not a projective plane, contradiction.

Hence, $d(x,y) = 2$. Let $z \in x^{\perp} \cap y^{\perp}$, and notice that $z \notin \ell \cup m$. We may assume $y^{\perp} \cap ax = \emptyset$ and $x^{\perp} \cap by = \emptyset$ by applying the previous argument to the point of $y^{\perp} \cap ax$ instead of x if $y^{\perp} \cap ax \neq \emptyset$ (and similarly for $x^{\perp} \cap by \neq \emptyset$). This yields that $x \notin S(a,y)$, $y \notin S(b,x)$. Apply now Lemma 4.1 to the 5-circuit x,a,b,y,z, to obtain $z \in a^{\perp} \cup b^{\perp}$. As this holds for any pair a,b with $a \in \ell$, $b \in m$, we have $z \in a_1^{\perp} \cap a_2^{\perp}$ for two distinct $a_i \in m$ (if necessary, after interchanging the role of ℓ and m). This implies $\ell \subseteq z^{\perp}$. Notice that $\{z\},m \subseteq a^{\perp} \cap y^{\perp}$ and that $z \notin m$ as $x^{\perp} \cap m = \emptyset$. Let b_1 be such that $\{b_1\} = z^{\perp} \cap m$. Now $z \in b_1^{\perp} \cap \ell^{\perp}$, so $z \in m^{\perp}$ by Lemma 3.7(iii), conflicting that $a^{\perp} \cap y^{\perp}$ is a generalized quadrangle. This contradiction ends the proof. □

4.3. COROLLARY.

(i) *If* x,y,z *form a clique in* P *and are not all three on a line, then* <x,y,z> *is a projective plane.*

(ii) *Each maximal singular subspace of* (P,L) *is a projective plane contained in at least one symp.*

(iii) *Axiom* (M1) *holds.*

PROOF. (i) According to Lemma 3.6(ii), <x,y,z> is a singular subspace of P, hence contained in a maximal singular subspace. But then the above lemma implies that <x,y,z> coincides with the maximal singular subspace.

(ii) Given a maximal singular subspace, take x,y,z not all three on a line. By Lemma 3.7(iii), $\{x,y,z\}^{\perp}$ is a maximal singular subspace, and by (i) this must be <x,y,z>. On the other hand, axiom (F2) leads to a point $u \in P \backslash z^{\perp}$ with $u \in (xy)^{\perp}$. Now u,z is a symp pair, so that S(u,z) is a symp containing x,y,z, hence <x,y,z>, and we are done.

(iii) Use the lemma and Corollary 3.10(ii). □

4.4. COROLLARY. *Let* $x \in P$ *and* $\ell \in L$ *satisfy* $d(x,\ell) = 2$. *Then* $x^{\perp} \cap \ell^{\perp}$ *contains at most one point. In other words,* (F8) *holds.*

PROOF. Suppose y,z are distinct points of $x^{\perp} \cap \ell^{\perp}$. Axiom (F3) applied to x,ℓ in $y^{\perp} \cap z^{\perp}$ yields that y,z are collinear. By the above lemma, $yz \cap \ell \neq \emptyset$, say $w \in \ell \cap yz$. Now $x \in (yz)^{\perp}$, so that $x \in w^{\perp}$ and $d(x,\ell) \leq 1$. This settles the corollary. □

4.5. LEMMA. *Suppose* S *is a symp in* P *and* $x \in P\backslash S$. *Then* $x^{\perp} \cap S$ *is either empty or a line.*

PROOF. Suppose $x \in y^{\perp}$ for some $y \in S$. If $z \in (x^{\perp} \cap S)\backslash y^{\perp}$, then $x \in y^{\perp} \cap z^{\perp} \subseteq S(y,z) = S$ by Corollary 3.10, conflicting $x \in P\backslash S$. So $x^{\perp} \cap S$ is a clique on y. In view of Corollary 4.3 we need only to show that $x^{\perp} \cap S$ contains a point distinct from y. Assume the contrary. Let $v \in S \cap \Gamma_2(y)$. By (F5), $d(v,x) \leq 2$, whence $d(v,x) = 2$. Let $z \in v^{\perp} \cap x^{\perp}$. Clearly $z \neq y$. If $u \in v^{\perp} \cap y^{\perp}$, then u,y,x,z,v is a 5-circuit with $d(u,x) = d(v,x) = d(v,y) = 2$. This 5-circuit is not in a symp, for else this symp would coincide with $S(v,y) = S$ so that $x \in S$. By Lemma 4.1, this implies $z \in u^{\perp} \cup y^{\perp}$. If $z \in y^{\perp}$, then $z \in \{x,y,v\}^{\perp} \subseteq x^{\perp} \cap S$ and we are done. We may therefore assume that $z \in u^{\perp}$ for any $u \in v^{\perp} \cap y^{\perp}$. Choose $u_1,u_2 \in v^{\perp} \cap y^{\perp}$ with $u_1 \notin u_2^{\perp}$. Then $z \in u_1^{\perp} \cap u_2^{\perp} \subseteq S$ by Corollary 3.10, whence $z \in x^{\perp} \cap S$, finishing the proof. □

4.6. LEMMA. *Let* $x \in P$, $\ell \in L$ *with* $d(x,\ell) = 2$ *and suppose* $x^{\perp} \cap \ell^{\perp} \neq \emptyset$. *Then there is exactly one point* y *in* ℓ *such that* x,y *is a symp pair.*

PROOF. We shall first establish existence. By Corollary 4.4 there is $c \in P$ with $\{c\} = x^{\perp} \cap \ell^{\perp}$. Thanks to (F2) we get $b \in \ell^{\perp}\backslash c^{\perp}$. According to axiom (F5), $d(x,b) \leq 2$. If $b = x$, then $d(x,\ell) = 1$ since $x,\ell \subseteq b^{\perp} \cap c^{\perp}$, contradiction. So $b \neq x$. But $b \notin x^{\perp}$ by Corollary 4.4. Hence $d(x,b) = 2$.

Let $w \in b^{\perp} \cap x^{\perp}$. Suppose there is no $y \in \ell$ such that x,y is a symp pair. Then x,c,y,b,w is a 5-circuit which is not contained in a symp for any $y \in \ell$. As $d(x,b) = d(b,c) = d(x,y) = 2$, Lemma 4.1 implies that $w \in c^{\perp} \cap y^{\perp}$. But if $w \in c^{\perp}$, then $\{w\},\ell \subseteq b^{\perp} \cap c^{\perp}$, so there is $u \in \ell$, with $w,c \in \{u,x\}^{\perp}$ and u,x is a symp pair, as desired.

Assume $w \notin c^{\perp}$. Then $w \in y^{\perp}$ for each $y \in \ell$, so $w \in x^{\perp} \cap \ell^{\perp}$, conflicting with Corollary 4.4. This shows that there is at least one $y \in \ell$ for which x,y is a symp pair.

Let us now suppose that y,z are distinct points of ℓ such that both x,y and x,z are symp pairs. Let c be as before and choose $a \in x^{\perp} \cap y^{\perp}$, $d \in x^{\perp} \cap z^{\perp}$ with $a,d \neq c$. Note that $a \neq d$, $d \notin y^{\perp}$, $a \notin z^{\perp}$ by Corollary 4.4. Now a,y,z,d,x is a 5-circuit not contained in a symp, for $d(x,yz) = 2$. Thus Lemma 4.1 yields $a \in d^{\perp}$. Take $v \in x^{\perp} \cap z^{\perp}\backslash d^{\perp}$, $v \neq c$. Then $a \in v^{\perp}$ by the same reasoning as above (with v instead of d), so that $S(x,z) = S(d,v) =$

$= S(a,z) = S(d,y) = S(x,y)$ by repeated application of Corollary 3.10(i). It follows that x,y,z are in one symp, which is absurd. This concludes the proof. □

4.7. LEMMA. *The intersection of two distinct symps is empty, a point, or a plane.*

PROOF. Let S,T be two distinct symps. In view of (M1) (see Corollary 4.3(iii)) it suffices to show that if $S \cap T$ contains a line ℓ, then $S \cap T$ is a plane. Choose $a,b \in \ell$ with $a \neq b$ and $x \in \Gamma_2(b) \cap \Gamma_1(a) \cap S$, $y \in \Gamma_2(a) \cap \Gamma_1(b) \cap T$. Note that $x \notin T$ and $y \notin S$. According to (F5), $d(x,y) \leq 2$. Note that $x \neq y$, because $x \notin \Gamma_2(a)$. If $d(w,y) = 1$ for a point $w \in xa$, then $x \in wa \subseteq S(a,y) = T$ and $S \cap T$ contains x,b while $x \notin b^{\perp}$, contradicting Corollary 3.10(ii). Thus $d(x,y) = 2$. Let $z \in x^{\perp} \cap y^{\perp}$ and consider the 5-circuit x,a,b,y,z. Clearly $z \notin ab$. As x,a,y are not in a symp, Lemma 4.1 implies that $z \in a^{\perp} \cap b^{\perp}$. Without harming generality we may assume $z \in a^{\perp}$. But then $z \in \{a,y\}^{\perp} \subseteq S(a,y) = T$. Notice that $a \in b^{\perp} \cap (xz)^{\perp}$ and $x,b \in S$, while $z,b \in T$. It follows from Lemma 4.6 applied to b and xz that $d(b,xz) = 1$. Let $w \in \Gamma_1(b) \cap xz$. If $w \neq z$, then $x \in wz \subseteq S(b,z) = T$, contradiction. The conclusion is that $z \in b^{\perp}$, so that $z \in b^{\perp} \cap x^{\perp} \subseteq S(b,x) = S$ and $z \in S \cap T$ as desired. □

Recall that for $w \in P$ the collection of lines (planes, symps, respectively) containing w is denoted by L_w, (V_w,S_w, respectively) and that $L(V_w)_w$ (and $L(S_w)_w$) denotes $\{L(M)_w \mid M \in V_w\}$ (and $\{L(S)_w \mid S \in S_w\}$, respectively).

4.8. LEMMA. *For each* $w \in P$, $(L_w, L(V_w)_w)$ *is a classical near hexagon whose quads are the members of* $L(S_w)_w$.

PROOF. $L_w \neq \emptyset$ by axiom (F2) applied to $w = x = y$. We verify the hypotheses of Lemma 3.3. Denote by d_w the distance function in the collinearity graph of $(L_w, L(V_w)_w)$. The lines of this incidence system are thick as the lines of (P,L) are thick. To begin, let $M \in V_w$ and $\ell \in L_w$. Then there is $m \in L\backslash L_w$ with $w \in m^{\perp}$ and $M = \langle m,w \rangle$ and $y \in \Gamma(w)$ such that $\ell = yw$. If $d_w(\ell,L(M)_w) = 0$, then $\ell \subseteq M$ and ℓ is the unique line in $L(M)_w$ nearest ℓ.

If $d_w(\ell L(M)_w) = 1$, then any line n in $L(M)_w$ nearest ℓ spans a plane with ℓ. So if n_1, n_2 are two such lines, $\ell \subseteq n_1^\perp \cap n_2^\perp = M$ by Corollary 4.3, contradicting $d_w(\ell, L(M)_w) = 1$.

Furthermore, $d_w(\ell, L(M)_w) > 1$ implies $d(y,m) = 2$ and by Lemma 4.6 there is a unique $x \in m$ such that $d_w(\ell, xw) = 2$.

So far, we have seen that $(L_w, L(V_w)_w)$ satisfies (N1) and is connected with diameter 2 or 3. We next claim that the quads of $(L_w, L(V_w)_w)$ are the members of $L(S_w)_w$. Let $\ell_1,\ell_2,\ell_3,\ell_4$ be a minimal circuit of $(L_w, L(V_w)_w)$. Choose $a_i \in \ell_i \backslash \{w\}$ for each i $(1 \le i \le 4)$. The quad containing ℓ_1, ℓ_3 is

$$Q(\ell_1,\ell_3) = \{\ell \in L_w \mid d_w(\ell, L(<\ell_{i+1},\ell_i>)_w) \le 1 \text{ for each } i\} =$$

$$= \{zw \mid z \in \Gamma(x) \text{ and } d(z, a_i a_{i+1}) \le 1 \text{ for each } i\} =$$

$$= \{zw \mid z \in \Gamma(x) \cap S(a_1,a_3) = L(S(a_1,a_3))_w$$

in view of Corollary 3.10(i). Conversely, if S is a symp on w, there is $u \in \Gamma_2(w)$ such that $u^\perp \cap w^\perp$ is a generalized quadrangle. Take a minimal 4-circuit a_1,a_2,a_3,a_4 in $u^\perp \cap w^\perp$ and define $\ell_i = a_i w$. Then

$$L(S)_w = L(S(a_1,a_2))_w = Q(\ell_1,\ell_3)$$

by the same argument as above. This establishes the claim that $L(S_w)_w$ is the collection of quads of $(L_w, L(V_w)_w)$.

Let $\ell, m \in L_w$ be two lines of distance 2 with respect to d_w, and let $n \in L_x$ satisfy $d_w(n,\ell) = d_w(n,m) = 1$. Taking $a_1 \in \ell$, $a_2 \in n$, $a_3 \in m$, all distinct from w, we get $a_1 \notin a_3^\perp$ and $n \subseteq a_1^\perp \cap a_3^\perp$ so that a_1, a_3 is a symp pair. It follows that ℓ, m are in the quad $L(S(a_1,a_3)_w)_w$.

Thus all hypotheses of Lemma 3.3 are satisfied. By Lemma 4.7 two quads never intersect in a point of $(L_w, L(V_w)_w)$ so that this incidence system is a classical near hexagon indeed. □

4.9. <u>LEMMA</u>. *Let* x ∈ P. *If there is exactly one symp* S *containing* x, *then* P = S.

<u>PROOF</u>. If $\ell \in L_x$, then $\ell \subseteq S$, for any line is contained in at least one symp.

Now let $y \in \Gamma(x)$. Then $y \in S$, as $yx \subseteq S$. We claim that S is the only symp on y. If the claim does not hold, there is a symp T containing y distinct from S. As $x^{\perp} \cap T \neq \emptyset$ and $x \in T$, Lemma 4.5 yields that $x^{\perp} \cap T$ is a line, say n, through y. For any $w \in n$, we have $w \in x^{\perp}$, so $w \in S$, whence $n \subseteq S \cap T$. According to Lemma 4.7, this implies that $M = S \cap T$ for some plane M. Take $v \in T\backslash M$ with $v \in n^{\perp}$. Then $n \subseteq x^{\perp} \cap v^{\perp}$, so x,v is a symp pair and $v \in S(x,v) = S$, contradicting $v \in T\backslash(S \cap T)$. This settles the claim that each neighbor y of x is contained in a unique symp, namely S. To end the proof of this lemma, use induction on the length of a path from an arbitrary point z to x to establish that $z \in S$. □

4.10. <u>COROLLARY</u>. $(P,\mathcal{L})$ *satisfies axiom* $(M3)_d$ *for* $d \in \{2,3\}$.

<u>PROOF</u>. In view of Lemma 4.8 we only need to verify that the diameter of $(\mathcal{L}_x, L(\mathcal{V}_x)_x)$ does not depend on $x \in P$. But this is a direct consequence of the above lemma. □

Note that if $d = 2$ in the above corollary, P consists of one symp only by Lemma 4.9, so that in fact P is a polar space of rank 3. Finally, $d = 3$ leads to $(P,\mathcal{L})$ being a metasymplectic space by Corollary 3.5.

5. CONNECTED METASYMPLECTIC SPACES ARE METAPOLAR

In this section, P is a set of points in which three families of subsets $\mathcal{L},\mathcal{V},\mathcal{S}$ are distinguished whose members are called lines, planes and symplecta, such that $P,\mathcal{L},\mathcal{V},\mathcal{S}$ form a metasymplectic space with thick lines and such that the incidence system $(P,\mathcal{L})$ is connected. According to Corollary 3.5, also $(M3)_3$ is satisfied. We shall verify axioms (F1),...,(F5) in order to finish the proof of Theorem 2.3. First of all, (F2) is immediate: since any line is in a symplecton by (M3) and since (F2) holds for polar spaces of rank 3, it holds for $(P,\mathcal{L})$ by (M2).

5.1. <u>LEMMA</u>.

(i) *Any plane together with the lines it contains is a projective plane.*

(ii) *Given* $M,N \in \mathcal{V}$ *such that* $M \cap N$ *is a line, then there is* $S \in \mathcal{S}$ *with* $M,N \subseteq S$.

(iii) *The intersection of two symplecta is either empty, a point, or a plane.*

(iv) *If* $S \in \mathcal{S}$ *and* $x,y \in P$ *with* $x \notin S$ *and* $y \in x^\perp \cap S$, *then there is a unique line* ℓ *in* S *on* y *such that* $\langle \ell,x\rangle$ *is a plane.*

PROOF. (i) By (M3), any plane is contained in a symplecton, so is a projective plane due to (M2).

(ii) Take $x \in M \cap N$ and set $\ell = M \cap N$. Then ℓ is a point of $(\mathcal{L}_x, L(\mathcal{V}_x)_x)$ on the two lines $L(M)_x, L(N)_x$, so by $(M3)_3$ there is a quad $L(S)_x$ for $S \in \mathcal{S}_x$ containing both $L(M)_x$ and $L(N)_x$. Now S is a symplecton as wanted.

(iii) Suppose S,T are distinct symplecta and ℓ is a line in $S \cap T$. By (M1), it suffices to prove that $S \cap T\backslash\ell \neq \emptyset$. For $x \in \ell$, the intersection of the quads $L(S)_x$ and $L(T)_x$ contains the point ℓ of $\mathcal{L}_x$ and therefore a line, say $L(M)_x$ for $M \in \mathcal{V}$ (see Lemma 3.3). It follows that $M \subseteq S \cap T$, so we are done.

(iv) Consider $(\mathcal{L}_y, L(\mathcal{V}_y)_y)$. It is a classical near hexagon by $(M3)_3$. By (N5) applied to the point, quad pair $\ell, L(S)_y$ there is a unique line m on y in S such that $\langle x,m\rangle$ is a plane. This ends the proof of the lemma. □

5.2. LEMMA. *If* $x,y,z \in P$ *satisfy* $z \in x^\perp \cap y^\perp$ *and if there is no symplecton containing all three* x,y,z, *then there are* $S,T \in \mathcal{S}$ *with* $x \in S$, $y \in T$ *such that* $S \cap T$ *is a plane containing* z.

PROOF. In the near hexagon $(\mathcal{L}_z, L(\mathcal{V}_z)_z)$, the lines xz and yz are points of distance 3. Let $u,v \in \Gamma_1(z)$ be such that xz,uz,vz,yz is a minimal path in $\mathcal{L}_z$. Then by Lemma 5.1(i) there is a symplecton S on x,u,z,v and a symplecton T on y,v,z,u.

Note that $\langle u,v,z\rangle$ is a plane as uz,vz are collinear in $\mathcal{L}_z$. Since $S \cap T$ contains u,v,z, we are done. □

5.3. LEMMA. *Let* $a,b,c \in P$ *be distinct points forming a clique. Then* $\langle a,b,c\rangle$ *is a line or a plane.*

PROOF. In view of axiom (M2), it suffices to show that all three points are in a symplecton. Assume there is no such symplecton. Then Lemma 5.2 yields a symplecton S containing b and a symplecton T containing c such that $M = S \cap T$ is a plane containing a. Notice that $b,c \notin M$. Lemma 5.1(iv) implies that there is a line ℓ through b in S such that $\langle \ell, c\rangle$ is a plane. As ℓ,M are both in S, axiom (M2) leads to a point z in M such that $\langle \ell, z\rangle$ is a plane. Now consider a symp U on z, c and ℓ, whose existence is guaranteed by Lemma 5.1(i). It intersects T in z and c, so $z \in c^{\perp}$ by axiom (M1). If z = a then $a,b,c \in U$, contradiction. But if $z \neq a$, then $\langle b, za\rangle$ and $\langle c, za\rangle$ are projective planes as they are contained in S and T, respectively, so that b,c,za are in a symplecton (cf. Lemma 5.1(ii)). This establishes the lemma. □

5.3. COROLLARY. (P,L) *satisfies* (F1).

5.4. LEMMA. *If* $S \in \mathcal{S}$ *and* $x,y \in S$ *are of mutual distance* 2, *then* $x^{\perp} \cap y^{\perp} \subseteq S$.

PROOF. Suppose $z \in x^{\perp} \cap y^{\perp}$ is not in S. Then by Lemma 5.1(iv), there is a line ℓ in $z^{\perp} \cap S$ on y such that $\langle \ell, z\rangle$ is a plane. Take $T \in \mathcal{S}$ containing this plane (such a T exists by M3). As $S \cap T$ contains ℓ, it is a plane, say M, according to Lemma 5.1(iii). Of course, $x \notin M$ as $d(x,y) = 2$. As a consequence of axiom M2 applied to S, the intersection $x^{\perp} \cap M$ is a line, say m. Now ℓ,m are both in the projective plane M, so they intersect, say, in u. As $\langle x,m\rangle$ and $\langle x,u,z\rangle$ are both planes (use Lemma 5.3) there is a quad $L(U)_x$ for $U \in \mathcal{S}$ containing both $(\langle x,m\rangle)_x$ and $L(\langle x,uz\rangle)_x$. Thus U is a symplecton containing x,m,z. It results that $U \cap T$ contains m and z. Moreover, U = T would imply $x,y \in T$ so that S = T by (M1), and $z \in S$ which is excluded. Thus U and T are distinct and intersect in a plane. Consequently, $\langle m,z\rangle$ is a plane contained in U, and $x \in \langle m,z\rangle^{\perp} \cap U$. This yields $x \in \langle m,z\rangle \subseteq T$ and T contains x,y. Again we are led to S = T and $z \in S$. This ends the proof. □

COROLLARY. *If* $S \in \mathcal{S}$ *and* $x \in P\backslash S$, *then* $x^{\perp} \cap S$ *is either empty or a line.*

PROOF. If $x^{\perp} \cap S$ is nonempty, then it is a clique by the above lemma, hence a singular subspace. In view of Lemma 5.1(iv), it cannot be a point or a plane. Therefore, S is as stated. □

5.6. COROLLARY. *If* $x,y,z \in P$ *form a clique and if they are not all three on a line, then* $\{x,y,z\}^{\perp} = \langle x,y,z\rangle$.

PROOF. Let $S \in \mathcal{S}$ contain x,y,z (use Lemma 5.3). If $u \in \{x,y,z\}^{\perp}$, then $u^{\perp} \cap S$

contains the plane <x,y,z>, so that $u \in S$ by the above corollary. This implies $u \in \langle x,y,z\rangle$ according to (M2). We have shown $\{x,y,z\}^\perp \subseteq \langle x,y,z\rangle$. The other inclusion is trivial. □

5.7. LEMMA. *If* $a \in P$ *and* $c \in \Gamma_2(a)$ *are such that* $a^\perp \cap c^\perp$ *has at least two points, then there is a unique symplecton containing both* a,c.

PROOF. The uniqueness results from (M1), so we need only prove the existence of a symplecton on a,c. If $a^\perp \cap c^\perp$ contains two distinct collinear points, this follows from Lemmas 5.1(ii) and 5.3. So we may assume that $a^\perp \cap c^\perp$ is a coclique. Let b,d be distinct points of $a^\perp \cap c^\perp$. In view of Lemma 5.4, it suffices to prove the existence of a symplecton on any three points of a,b,c,d. Using Lemma 5.2, we get symplecton A and C containing a and c respectively such that $M = A \cap C$ is a plane on d. By the above remark, we need only consider the case where $b \notin A \cup C$. Then $\ell = b^\perp \cap A$ and $m = b^\perp \cap C$ are lines on a and c respectively by Corollary 5.5. Applying (M2) to A,C, we obtain points $z_1, z_2 \in M$ with $z_1 \in \ell^\perp$ and $z_2 \in m^\perp$. Moreover, by (P2) for C and A, there are $y_1 \in m \cap z_1^\perp$ and $y_2 \in \ell \cap z_2^\perp$. Notice that $z_1 \in b^\perp$ would imply the existence of a quad in L_a on ab, az_1, ad so that we are done. Similarly for $z_2 \in b^\perp$. Thus we may restrict attention to the case where $d(b,z_2) = d(b,z_1) = 2$. Now let S_1 be the symplecton containing b,z_1,ℓ and let S_2 be the symplecton containing b,z_2,m. Using Lemma 5.4, we get $y_1 \in S_1$ and $y_2 \in S_2$. But $y_1 \in m \subseteq S_2$ and similarly $y_2 \in S_1$, so that $y_1, y_2 \in S_1 \cap S_2$. Hence $y_1 \in y_2^\perp$. Since $y_1 = y_2$ would lead to the existence of a symplecton on a, by_1 and c, we may assume $y_1 \neq y_2$. Also, we may assume $z_2 \notin S_1$, for else $d \in z_2^\perp \cap a^\perp \subseteq S_1$. Now $z_2^\perp \cap S_1$ contains y_1, y_2, z_1 so $z_1 \in y_1y_2$ by Corollary 5.5. But then, again, $z_1 \in b^\perp$ and the Lemma is proved. □

5.8. COROLLARY. *Axioms* (F3) *and* (F5) *hold for* (P,L).

PROOF. Suppose $x \in P$ and $y \in \Gamma_2(x)$ are such that $x^\perp \cap y^\perp$ contains at least two points. Then they are contained in a symplecton S as we have just seen, so $x^\perp \cap y^\perp$ is a generalized quadrangle with thick lines. This proves (F3). Now let $z \in y^\perp$. If $z \in S$, then clearly $d(x,z) \leq 2$. Suppose therefore $z \notin S$. Since $y \in z^\perp \cap S$, there is a line ℓ in $z^\perp \cap S$ by Corollary 5.5. Now there is $w \in x^\perp \cap \ell$ by axiom (P2) for S, so that $w \in x^\perp \cap z^\perp$. This shows that

$d(x,z) \leq 2$, whence (F5). □

According to Lemma 5.4, a point is contained in a symplecton if and only if it has two mutually noncollinear neighbors in the symplecton. The following lemma provides the line analogue of this criterion of containment in a symplecton.

5.9. LEMMA. *Given a symplecton* S *and two distinct collinear points* x,y *for which there are distinct mutually noncollinear* $u,v \in S$ *with* $u \in x^{\perp} \cap S\backslash y^{\perp}$ *and* $v \in y^{\perp} \cap S\backslash x^{\perp}$, *then* $x,y \in S$.

PROOF. If $xy \cap S \neq \emptyset$, then Lemma 5.4 suffices for the proof. Assume $xy \subseteq P\backslash S$. Set $\ell = x^{\perp} \cap S$ and $m = y^{\perp} \cap S$. By Corollary 5.5, ℓ,m are lines on u,v respectively. Thus $\ell \neq m$. If $z \in \ell \cap m$, then $L(S)_z$ is a quad in L_z, and $L(M)_z$ where M is the plane $<xy,z>$, is a line of $(L_z,L(V_z)_z)$ such that $L(M)_z$ has xz and yz collinear with the noncollinear points uz and vz of the quad. As a consequence of (N6), $L(M)_z$ is in the quad $L(S)_z$, so $x,y \in S$ as wanted. Therefore we may assume $\ell \cap m = \emptyset$ for the rest of the proof. Let $a \in u^{\perp} \cap m$. Now ua is a point in L_a of distance 2 from both va and ya (use Lemma 5, to derive that u,a,x,y are in a symplecton). Therefore, there is a line n through a in $<m,y> \cap u^{\perp}$. Note that $y \notin n$ because $y \notin u^{\perp}$. Due to Lemma 5.1(ii), there is a symplecton, say T, on u,n,y. But $x^{\perp} \cap T$ contains u and y, so $x \in T$. Furthermore, $v \in <m,y> \subseteq T$ and for any $w \in v^{\perp} \cap \ell$ we have $x,v \in w^{\perp} \cap T$, so that w and therefore ℓ is in T. We conclude that $S = T$ and that x,y are in S. □

5.10. COROLLARY. *There is no minimal 5-circuit in* P *three of whose points are in a symplecton.*

PROOF. Let S be a symplecton and let a_1,a_2,a_3,a_4,a_5 be a minimal 5-circuit with three points in S. If three consecutive points of the circuit are in S, the above lemma, shows that there is a minimal 5-circuit in the polar space S, which is absurd. So the proof is reduced to the case where (up to a shift of indices modulo 5) a_1,a_2,a_4 are members of S. But then $a_4^{\perp} \cap a_1a_2 \neq \emptyset$ by (M2) contradicting the minimality of the circuit. This settles the lemma. □

5.11. LEMMA. (P,L) *satisfies* (F4).

PROOF. Assume a_1,a_2,a_3,a_4,a_5 is a minimal 5-circuit. By Corollary 5.10 there is no symplecton on a_1,a_2,a_3. According to Lemma 5.2 there are symplecta S,T such that $a_1 \in S$ and $a_3 \in T$ while $M = S \cap T$ is a plane containing a_2. Again by Corollary 5.10, $a_4,a_5 \notin S \cup T$. Set $\ell = a_5^\perp \cap S$ and $m = a_4^\perp \cap T$. Clearly, $a_1 \in \ell$ and $a_3 \in m$. Since M,ℓ are in the polar space S, there is a point $z \in \ell^\perp \cap M$. As z,m are in T, we have $w \in z^\perp \cap m$. Of course, $z \neq a_1,a_3$ as $a_1,a_3 \notin M$. Notice that $z \notin \ell$ (and $z \notin m$) for else there would be a symplecton on a_2,ℓ,a_5 (or a_2,m,a_4, respectively) by Lemma 5.1(ii), contradicting Corollary 5.10. Thus $z \notin a_5^\perp \cup a_4^\perp$. Denote by U the symplecton on z,ℓ,a_5. Now $z \in w^\perp \cap U \backslash a_4^\perp$ and $a_5 \in a_4^\perp \cap U \backslash z^\perp$, so wa_4 is contained in U by Lemma 5.9. This, however, leads to the final contradiction (Corollary 5.10) that a_4,a_5,a_1 are in the symplecton U. □

Having verified axiom (F1) in Corollary 5.3, (F2) in the beginning of this section, (F3) and (F5) in Corollary 5.8, and (F4) in Lemma 5.11, we conclude that the points and lines of any connected metasymplectic space with thick lines form a metapolar space. As the same is obviously true for a polar space of rank 3 with thick lines, we have established the converse of the statement in Section 4, and hence finished the proof of Theorem 2.3.

6. REFERENCES

[1] BUEKENHOUT, F. & E.E. SHULT, *On the foundations of polar geometry*, Geometriae Dedicata 3 (1974), 155-170.

[2] CAMERON, P.J., *Dual polar spaces*, to appear.

[3] COHEN, A.M., *Geometries originating from certain distance-regular graphs*, in: Finite Geometries and Designs, LMS Lecture Note 49 (Cameron, Hirschfeld, Hughes eds), London 1981.

[4] COOPERSTEIN, B.N., *Some geometries associated with parabolic representations of groups of Lie type*, Can. J. Math. 28 (1976), 1021-1031.

[5] COOPERSTEIN, B.N., *A characterization of some Lie incidence structures*, Geometriae Dedicata 6 (1977), 205-258.

[6] FREUDENTHAL, H., *Beziehungen der E_7 und E_8 zur Oktavenebene*, I-XI, Proc. Kon. Ned. Akad. Wet. A57 (1954), 218-230, 363-368; A58 (1955), 151-157, 277-285; A62 (1959), 165-201, 447-474; A66 (1963), 457-487 (= Indag. Math., 16,17,21,25).

[7] SHULT, E.E. & A. YANUSHKA, *Near n-gons and line systems*, Geometriae Dedicata, 9 (1980), 1-72.

[8] TITS, J., *Buildings of spherical type and finite BN-pairs*, Lecture Notes in Mathematics 386, Springer, 1974.